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Frequency shifts in a single-mode laser

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Abstract. The semi-classical laser equations are analysed outside the rotating-wave approximation and a new exact steady-state solution for stationary but inhomogeneously broadened atoms is obtained. Results applicable to any solid-state laser are derived and these are also relevant to a gas laser working near threshold or at very high intensities. The resonance frequency exhibits a Bloch–Siegert shift. New results in the theory of frequency pulling and pushing are obtained.

1. Introduction

Several attempts have been made to improve upon the semi-classical theory of a laser as first presented by Lamb (1964). Uehara and Shimoda (1965) carried the perturbation expansion of the polarization as a power series in the field amplitude to fifth order whilst Culshaw (1967) extended the calculation to even higher orders. However, series expansions beyond third order are rather cumbersome and contain little physical information. Moreover the validity of such a series is limited by the requirement that the intensity remains so small as to make the dimensionless parameter I,

$$I \equiv \frac{\frac{1}{2}T_1 T_2 p^2 \mathscr{E}_0^2}{\hbar^2}, \tag{1.1}$$

much less than unity. Here p is the dipole matrix element between the two states of interest, $\mathscr{E}_0$ is the field amplitude and the times T_1 and T_2 are respectively measures of the rate at which the excess population decays to the time independent value established by a pumping mechanism competing with damping and the rate at which phase correlation between atoms is destroyed. Holt (1970) has estimated that the perturbation series starts breaking down when I exceeds 0·05. Typically for ruby or neodymium $T_1 \sim 10^{-3}$ s and $T_2 \sim 10^{-11}$ s† however, and since $p^2 \sim 10^{-39}$ cgs units, the perturbation series expansion is valid only for intensities $\mathscr{E}_0^2 < 10^{-2}$. Unfortunately this inequality is not satisfied for intensities at which these lasers are normally operated.

For a single-mode laser the theory has been extended to higher intensities (Greenstein 1968, Stenholm and Lamb 1969, Stenholm 1970, 1971, Holt 1970; an exhaustive list of references appears in Stenholm 1971). However, the calculation generally becomes so involved that the search for an analytical solution has to be abandoned except for

† Statz (1967) quotes the following values for T_1 and T_2:

	T_1	T_2
Ruby	5×10^{-3} s	$1/2{\cdot}4\times10^{-11}$ s
Nd glass	3×10^{-4} s	$1/2{\cdot}4\times10^{-11}$ s.

special cases. The simplest case to treat is undoubtedly that of stationary atoms. Lamb (1964) obtained an exact analytical solution for this case and pointed out that the effect of atomic motion in a gas cannot be properly taken into account by simply assuming that the only effect of atomic motion is to Doppler shift the atomic resonance frequency.

All calculations to date have used the rotating-wave approximation which drops *all* rapidly oscillating terms. The purpose of this paper is to investigate the role of this approximation in laser theory. One intuitively expects that the retention of the 'counter-rotating terms' in the theory will result in a shift of the resonance (ie Bloch–Siegert shift) and may have significance in the theory of frequency pulling and pushing as well. These expectations are borne out by a steady-state solution valid for stationary atoms which we report in this paper. However, the Bloch–Siegert shift for an atom in a cavity turns out to be only three quarters of its value for a free atom. We also obtain a generalized expression for frequency pulling and find, in particular, that at high intensities the oscillator frequency coincides, for all practical purposes, with the cavity resonance frequency. Throughout we merely assume that the atomic resonance frequency ω_s is inhomogeneously broadened with the help of a distribution function $g(\omega_s)$. This procedure should be adequate for a solid-state laser. The results should be applicable to the gas laser near the threshold conditions (Lamb 1964) and they should also apply at high intensities where power broadening makes the effect of atomic motion negligible (Stenholm 1970).

In § 2 which follows we present the semi-classical laser equations and their steady-state solution. In order to achieve comparison with previous work the equations we use are essentially those of Lamb although we adopt the Bloch equation form. A generalized version of these equations appears in Bullough *et al* (1974). In § 3 we use the solution to discuss frequency pulling and pushing in the course of which we make a direct but brief comparison with standard results in refractive index theory. These results are also relevant to the theory of partial mode locking (Picard and Schweitzer 1969) and pulse compression (Treacy 1969).

2. Equations of motion

Our formalism is that of Lamb (1964) except that we shall use the Bloch vector $\boldsymbol{r} \equiv (r_1, r_2, r_3)$ instead of individual elements of the atomic density matrix. The components r_1 and r_3 of $\boldsymbol{r}$ have immediate physical significance: pr_1 is the atomic dipole, r_3 is the inversion. We shall here introduce two independent empirical time constants T_2 and T_1 to describe their decay.

We assume the electric field to be a standing wave of the form

$$E(z, t) \equiv A(t)U(z) \tag{2.1}$$

where $U(z) = \sin kz$; $k = \Omega c^{-1}$; and Ω is a cavity eigenfrequency. We define $P(t)$ to be the space Fourier transform of the polarization $P(z, t)$

$$P(t) = \frac{2}{L}\int_0^L P(z, t) \sin kz \, \mathrm{d}z. \tag{2.2}$$

The cavity is one dimensional, of length L.

We assume that a steady-state purely harmonic solution exists, ie

$$A(t) = \mathscr{E}_0 \cos(\nu t + \phi) \tag{2.3a}$$

$$P(t) = C \cos(\nu t + \phi) + S \sin(\nu t + \phi). \tag{2.3b}$$

For a single mode we can choose $\phi \equiv 0$. The driven Maxwell wave equation

$$\nabla^2 \boldsymbol{E} - \frac{4\pi\sigma}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} = \frac{4\pi}{c^2}\frac{\partial^2 \boldsymbol{P}}{\partial t^2} \tag{2.4}$$

where the 'conductivity' σ accounts for the losses from the cavity, reduces to

$$(\nu^2 - \Omega^2)\mathscr{E}_0 = -4\pi\nu^2 C \tag{2.5a}$$

$$\tfrac{1}{2}\nu Q^{-1}\mathscr{E}_0 = -2\pi\nu S \tag{2.5b}$$

where Q is the quality factor of the cavity.

The medium is assumed to consist of two-level atoms: the upper states are $|s\rangle$ of energy $\frac{1}{2}\hbar\omega_s$; the lower states are $|0\rangle$ with energy $-\frac{1}{2}\hbar\omega_s$. The components of the Bloch vector are defined by

$$\begin{aligned} r_1 &= \rho_{s0} + \rho_{0s} \\ r_2 &= \mathrm{i}(\rho_{s0} - \rho_{0s}) \\ r_3 &= \rho_{ss} - \rho_{00}; \end{aligned} \tag{2.6}$$

the $\rho_{\alpha\beta}$ are elements of the density matrix of a 'typical atom' of frequency ω_s: r_1, r_2 and r_3 depend on z, t and ω_s. The polarization of the medium is

$$P(z, t) = np \int_0^\infty g(\omega_s) r_1(z, t; \omega_s)\, \mathrm{d}\omega_s. \tag{2.7}$$

The function $g(\omega_s)$ is the normalized frequency distribution describing the inhomogeneous broadening: n is the total number density of atoms of whatever frequency and state of excitation.

With empirically added damping the Bloch vector satisfies

$$\frac{\partial r_1}{\partial t} = -\omega_s r_2 - \frac{r_1}{T_2} \tag{2.8a}$$

$$\frac{\partial r_2}{\partial t} = \omega_s r_1 - w r_3 - \frac{r_2}{T_2} \tag{2.8b}$$

$$\frac{\partial r_3}{\partial t} = w r_2 - \frac{r_3 - r_3(z, t_0)}{T_1} \tag{2.8c}$$

where

$$w(z, t) \equiv -2p\hbar^{-1}E(z, t). \tag{2.8d}$$

Here the quantity $nr_3(z, t_0)$ plays the same role as the 'excitation density' in Lamb's formalism and conceals a rate Λ_s (say) of pumping of atoms into upper states $|s\rangle$ and a rate Λ_0 of removal to lower states. The Bloch equations (2.8) are equivalent to the set of equations

$$\frac{\partial^2 r_1}{\partial t^2}+\frac{2}{T_2}\frac{\partial r_1}{\partial t}+\omega_s^2 r_1 = \omega_s w r_3 \tag{2.9a}$$

$$\frac{\partial r_1}{\partial t} = -\omega_s r_2-\frac{r_1}{T_2} \tag{2.9b}$$

$$\frac{\partial r_3}{\partial t} = w r_2-\frac{r_3-r_3(z,t_0)}{T_1}. \tag{2.9c}$$

We want to find a steady-state solution of the coupled equations (2.5), (2.7) and (2.9).

We assume a solution for r_1 in the form

$$r_1 = C_a(z,\omega_s)\cos vt+S_a(z,\omega_s)\sin vt. \tag{2.10}$$

The field $E(z,t)$ is as assumed in (2.1). The subscript a refers to the atomic quantities depending on z and labelled by frequencies ω_s. The corresponding macroscopic quantities are

$$C = \frac{2}{L}np\int_0^L\int_0^\infty C_a(z,\omega_s)g(\omega_s)\sin kz\,dz\,d\omega_s \tag{2.11}$$

and similarly for S. The assumptions (2.1) and (2.10) in (2.9) mean that

$$\begin{aligned}\frac{\partial^2 r_1}{\partial t^2}&+\frac{2}{T_2}\frac{\partial r_1}{\partial t}+\omega_s^2 r_1\\ &= -2w_0\omega_s\cos vt\left\{U(z)r_3(z,t_0)+w_0\omega_s^{-1}T_1\left(vS_a+\frac{C_a}{T_2}\right)U^2(z)\right.\\ &\quad+w_0\omega_s^{-1}U^2(z)\left[\left(\frac{S_a}{2}+\frac{C_a}{2T_2v}-\frac{C_a}{4T_1v}\right)\sin 2vt\right.\\ &\quad\left.\left.+\left(\frac{C_a}{2}+\frac{S_a}{4T_1v}-\frac{S_a}{2T_2v}\right)\cos 2vt\right]\right\}\end{aligned} \tag{2.12}$$

Here w_0 is the 'Rabi frequency': $w_0 = p\mathscr{E}_0/\hbar$. We will retain only those terms on the right-hand side which can produce resonance with the field although it is easy to see how to extend the theory to include steady-state harmonics: this is not equivalent to the rotating-wave approximation which has not been made. We can also consistently neglect C_a/T compared with vS_a. The result is

$$\begin{aligned}\frac{\partial^2 r_1}{\partial t^2}&+\frac{2}{T_2}\frac{\partial r_1}{\partial t}+\omega_s^2 r\\ &= -2w_0\left[U(z)r_3(z,t_0)\omega_s+w_0T_1vS_a(z,\omega_s)U^2(z)\right.\\ &\quad\left.+\tfrac{1}{2}w_0U^2(z)\left(\frac{C_a(z,\omega_s)}{2}+\frac{S_a(z,\omega_s)}{4T_1v}-\frac{S_a(z,\omega_s)}{2T_2v}\right)\right]\cos vt\\ &\quad-\tfrac{1}{2}w_0^2U^2(z)S_a(z,\omega_s)\sin vt\end{aligned} \tag{2.13}$$

(in which $U(z) \equiv \sin kz$). We multiply both sides by $(2/L)\sin kz$ and integrate on z. We denote

$$\frac{2}{L}\int_0^L r_1(z,t;\omega_s)\sin kz\,dz \equiv r_f; \tag{2.14}$$

and assume

$$\frac{2}{L}\int_0^L \begin{pmatrix} C_a(z,\omega_s) \\ S_a(z,\omega_s)\end{pmatrix} \sin 3kz\,dz \ll \frac{2}{L}\int_0^L \begin{pmatrix} C_a(z,\omega_s) \\ S_a(z,\omega_s)\end{pmatrix} \sin kz\,dz \tag{2.15}$$

and

$$\frac{2}{L}\int_0^L r_3(z,t_0)\cos 2kz\,dz \ll \frac{2}{L}\int_0^L r_3(z,t_0)\,dz. \tag{2.16}$$

Assumption (2.16) is equivalent to the assumption of negligible spatial variations in the steady-state inversion density and (2.15) is consistent with the assumption of oscillations in a single mode. We now find

$$\begin{aligned}\frac{\partial^2 r_f}{\partial t^2}+\frac{2}{T_2}\frac{\partial r_f}{\partial t}+\omega_s^2 r_f &= -2w_0\left[\bar r_3(t_0)\omega_s+\tfrac{3}{4}T_1 v S_{a'} w_0+\tfrac{3}{8}w_0\left(\frac{C_{a'}}{2}+\frac{S_{a'}}{4T_1 v}-\frac{S_{a'}}{2T_2 v}\right)\right]\cos vt \\ &\quad -\tfrac{3}{8}w_0^2 S_{a'}\sin vt \end{aligned}\tag{2.17}$$

with

$$\begin{pmatrix} C_{a'} \\ S_{a'}\end{pmatrix} \equiv \frac{2}{L}\int_0^L \begin{pmatrix} C_a(z,\omega_s) \\ S_a(z,\omega_s)\end{pmatrix}\sin kz\,dz \tag{2.18a}$$

and

$$\bar r_3(t_0) \equiv \frac{1}{L}\int_0^L r_3(z,t_0)\,dz. \tag{2.18b}$$

The quantity $n\bar r_3(t_0)$ is the average population inversion density before the start of oscillations: we denote it by R_0.

We can now substitute for r_f in (2.17) from (2.10) and compare coefficients of $\sin vt$ and $\cos vt$. We have finally

$$S_{a'} = \frac{-4vT_2^{-1}w_0\omega_s\bar r_3(t_0)}{(\omega_s^2-v^2+\frac{3}{8}w_0^2)^2+4v^2T_2^{-2}+3v^2w_0^2T_1T_2^{-1}} \tag{2.19a}$$

and

$$C_{a'} = \frac{-2\omega_s w_0\bar r_3(t_0)(\omega_s^2-v^2+\frac{3}{8}w_0^2)}{(\omega_s^2-v^2+\frac{3}{8}w_0^2)^2+4v^2T_2^{-2}+3v^2w_0^2T_1T_2^{-1}}. \tag{2.19b}$$

The structure and, to some extent, the form of the results (2.19) compares with the results, equations (52)–(56) of Stenholm and Lamb (1969). If one replaces $\sin kz$ by its average value of $\frac{1}{2}$ then their expressions for $S_{a'}$ and $C_{a'}$ read in the present notation

$$S_{a'} = \frac{-w_0T_2^{-1}\bar r_3(t_0)}{(\omega_s-v)^2+T_2^{-2}+\frac{1}{2}w_0^2T_1T_2^{-1}} \tag{2.20a}$$

and

$$C_{a'} = (\omega_s - v)T_2 S_{a'}. \tag{2.20b}$$

Equations (2.19) differ from (2.20) in two important respects. First: the resonance is shifted by an amount $3w_0^2/16v$ to first order in w_0^2/v^2. This is immediately recognizable as the leading term in the power series development of the Bloch–Siegert shift (Ahmad and Bullough 1974). Notice that the cavity has reduced this shift by 25%. Also this shift does not depend on T_1 or T_2. This conclusion is supported by recent experiments of Arimondo and Moruzzi (1973). Second: the expressions (2.19) resonate not only at $v \simeq \omega_s$ but also at $v \simeq -\omega_s$. Equations (2.19) will go over into (2.20) provided we replace w_0^2 in the last term of the denominator by $\frac{2}{3}w_0^2$, ignore the Bloch–Siegert shift and set $\omega_s + v = 2\omega_s = 2v$. We shall show below that the 'negative frequency terms' play an important role in the theory.

The procedure used in this section for solving the Bloch equations is formally equivalent to the one adopted by Arimondo (1968). He expanded r_1, r_2 and r_3 in Fourier series, substituted the series in the Bloch equations (with $T_1 = T_2$ however) and solved the resulting simultaneous equations, six in number, for the coefficients of $\cos vt$ and $\sin vt$. However, the present method appears to be more suitable for the problem at hand.

3. Discussion of the steady-state solution

The spread in values of ω_s is typically of the order of 10^{12} Hz. Therefore we can safely assume

$$|\omega_s^2 - v^2| \ll v^2 T_1 T_2^{-1}. \tag{3.1}$$

Now if the field is weak enough

$$v^2 T_1 T_2^{-1} w_0^2 \ll v^2 T_2^{-2}. \tag{3.2}$$

Condition (3.2) is precisely the condition $I \ll 1$ however, so that $\mathscr{E}_0^2 \ll 10^{-2}$ cgs units and the results apply only near threshold. If (3.2) holds we have

$$pS_{a'} \simeq \frac{-(4v/T_2)(p^2\mathscr{E}_0/\hbar)\omega_s \bar{r}_3(t_0)}{(\omega_s^2 - v^2)^2 + (4v^2/T_2^2)} \tag{3.3a}$$

$$pC_{a'} \simeq -\frac{2\omega_s(\omega_s^2 - v^2)(p^2\mathscr{E}_0/\hbar)\bar{r}_3(t_0)}{(\omega_s^2 - v^2)^2 + (4v^2/T_2^2)}. \tag{3.3b}$$

These are the real and imaginary parts of the atomic polarizability

$$\alpha(v) = -\frac{2\omega_s p^2 \hbar^{-1} \bar{r}_3(t_0)}{\omega_s^2 - v^2 + \mathrm{i}(2v/T_2)} \tag{3.4}$$

stimulated linearly by the field $\mathscr{E}_0 \cos vt$.

The many-body problem in the linear approximation equivalent to (3.4) has been largely solved for two cases: these are the rigid crystal and the ensemble averaged 'molecular fluid' (cf, eg, Obada and Bullough 1969, Hopfield 1958, Bullough and Thompson 1970, Bullough *et al* 1968, Bullough and Hynne 1968). It is assumed there that $\bar{r}_3(t_0) = -1$ (the attenuator) but the calculation is unchanged for other values of $r_3(z, t_0)$ providing this does not, in fact, vary with z. The key results in amorphous systems are that $\alpha(v)$ should be replaced by the Lorentz and local-field corrected result

$$\alpha_L(v) = \frac{-2\omega_s p^2 \hbar^{-1} \bar{r}_3(t_0)}{\omega_s^2 - v^2 + \frac{8}{3}\pi n \omega_s p^2 \hbar^{-1} \bar{r}_3(t_0) + nJ(v)} \tag{3.5}$$

and that in consequence the wavenumber k of the mode is not $\omega_s c^{-1}$ on resonance; $J(v)$ is a complex-valued function of v so that there is a change in the damping, a shift of the resonance, and a possible change in the character of the singularities of $\alpha_L(v)$ at the zeros of the denominator. Essentially the same features apply to crystals: the Lorentz field correction can apparently be inadequate even in cubic crystals just in the resonance region (Bullough and Thompson 1970).

We shall here first pursue the argument with (3.4) rather than (3.5) keeping the empirical inhomogeneous broadening function $g(\omega_s)$ as a description of the microscopic broadening mechanisms some of which are otherwise described by $J(v)$. We prove $\Omega = ck \neq \omega_s$ 'on resonance' (where $v = \omega_0$) solely because (3.4) contains negative frequency terms resonating at $v \sim -\omega_s$: the relative shift is small but potentially significant in laser theory.

If the width $\Delta\omega_0$ of $g(\omega_s)$ is much greater than T_2^{-1} (as is usually the case) we can replace $g(\omega_s)$ by its value at the line centre, $g(\omega_0)$ and integrate to get

$$S = -p^2 \mathscr{E}_0 \hbar^{-1} R_0 g(\omega_0) \left[\tan^{-1}\left(\frac{\omega_s^2 - v^2}{2vT_2^{-1}}\right)\right]_0^{\infty} = -\pi p^2 \mathscr{E}_0 \hbar^{-1} R_0 g(\omega_0).$$

Using this in (2.5*b*) we find the threshold condition

$$Q^{-1} = 4\pi^2 p^2 \hbar^{-1} R_0 g(\omega_0). \tag{3.6}$$

The comparable result for C when substituted in (2.5*a*) yields

$$v^2 - \Omega^2 = \frac{1}{2\pi}\frac{v^2}{Q}\left[\ln\left((\omega_s^2 - v^2)^2 + \frac{4v^2}{T_2^2}\right)\right]_0^{\infty} \tag{3.7}$$

after making use of (3.6). The result (3.7) should be better approximated by restricting the contribution to the integral over ω_s to a range $\omega_0 - \Delta\omega_0 < \omega_s < \omega_0 + \Delta\omega_0$ with $\Delta\omega_0$ the linewidth. Replacement of the limits for ω_s in (3.7) to the ends of this range yields

$$v^2 - \Omega^2 = \frac{1}{2\pi}\frac{v^2}{Q}\ln\left(\frac{[\omega_0^2 - v^2 + (\Delta\omega_0)^2 + 2(\Delta\omega_0)\omega_0]^2 + (4v^2/T_2^2)}{[\omega_0^2 - v^2 + (\Delta\omega_0)^2 - 2(\Delta\omega_0)\omega_0]^2 + (4v^2/T_2^2)}\right). \tag{3.8}$$

For large detuning we can neglect $(\Delta\omega_0)^2$ in (3.8) and obtain

$$v^2 - \Omega^2 \simeq \frac{1}{2\pi}\frac{v^2}{Q}\ln\left(\frac{(\omega_0 - v + \Delta\omega_0)^2 + (1/T_2^2)}{(\omega_0 - v - \Delta\omega_0)^2 + (1/T_2^2)}\right). \tag{3.9}$$

We use $\omega_0 + v \simeq 2v$ which is equivalent to the rotating-wave approximation. Equation (3.9) then yields to first order in $(\omega_0 - v)/\Delta\omega_0$ the familiar form (Lamb 1964, equation (67))

$$\frac{v - \Omega}{\omega_0 - v} \simeq \frac{v}{\pi(\Delta\omega_0)Q}. \tag{3.10}$$

However, this result uses

$$|\omega_0^2 - v^2| \gg (\Delta\omega_0)^2 \tag{3.11a}$$

(large detuning) whereas, close to resonance

$$|\omega_0^2 - v^2| \ll (\Delta\omega_0)^2 \tag{3.11b}$$

so that

$$v^2 - \Omega^2 \simeq \frac{1}{2\pi}\frac{v^2}{Q}\ln\left(1+\frac{2\Delta\omega_0}{\omega_0}\right)$$

or

$$v - \Omega \simeq \frac{\Delta\omega_0}{2\pi Q}. \tag{3.12}$$

With Q of the order of 10^7 and the inhomogeneous linewidth, $\Delta\omega_0$, exceeding 10^{12} Hz, as for example, for a neodymium-glass laser, the frequency difference $v-\Omega$ may approach 10^5 Hz. This frequency pulling differs from the result of the rotating-wave approximation: in this approximation (3.10) is supposed to apply on resonance also and $v = \omega_0$ implies $v = \Omega$ as well. Since at high intensities $v-\Omega$ is very small (see equation (3.19) below) it follows that the pulsed output from a neodymium-glass laser will be carrier-frequency modulated. This modulation, or chirp, as it is generally called, is of the order mentioned above and is essential for the pulse compression experiments (Treacy 1969). It is also clear from (3.12) that the chirp for most other lasers is negligibly small.

The physical significance of (3.12) becomes clear by noting that for sufficiently small $(\Delta\omega_0)^2/\omega_0^2$ equation (3.8) reduces to (3.10) even when (3.11*b*) is satisfied provided we replace ω_0 by ω_0' where

$$\omega_0' \simeq \omega_0 + \frac{(\Delta\omega_0)^2}{\omega_0}. \tag{3.13}$$

This suggests that the oscillator frequency is pulled towards a new line centre ω_0'. The shift for a gas laser, He–Ne for example ($\Delta\omega_0 \sim 10^9$ Hz) will be only about a thousand hertz. We must point out that (3.13) does not imply that ω_0' will be the most favoured frequency, ie one which experiences highest gain.

3.1. Partial mode locking

Most theories of self-locking of modes in a multi-mode laser assume that all adjacent modes are equally spaced (cf, eg, Statz *et al* 1967). According to Statz (1967) this situation is unlikely to prevail in a material with a large inhomogeneously broadened line as for example in a neodymium-glass laser and locking should not occur. This is contradicted by experiments where ultra-short pulses have been observed in the output of practically any solid-state laser (see, eg, Bass and Woodward 1968). This discrepancy may be qualitatively understood by noting that in a material with a large inhomogeneous linewidth, $\Delta\omega_0$, inequality (3.11*b*) may be satisfied for a large number of modes. Frequencies of all such modes will be equally spaced since the right-hand side of (3.12) does not depend on the detuning $\omega_0 - v$. These modes will be available for self-locking. However, there may also be modes oscillating at frequencies far removed from the line centre so that (3.11*a*) is satisfied. In this case (3.9) applies and the oscillation frequencies *do* depend on the detuning. Statz' analysis may apply in this region and these modes will resist being locked. This argument emphasizes the importance of the inhomogeneous linewidth, $\Delta\omega_0$, to the theory of partial-mode locking (Harrach 1968, Picard and Schweitzer 1969).

3.2. High-intensity limit

We now examine the extreme case when the intensity is so high that only the last term in the denominator of (2.19*a*) is significant. This implies

$$I \gg (T_2 \Delta\omega_0)^2 \tag{3.14}$$

where I is defined by (1.1). Physically condition (3.14) means that the atomic line is power broadened to such an extent that it dominates the inhomogeneous broadening. This condition is unlikely to be satisfied for gas lasers but may be met for high power solid-state lasers. In this case

$$S_{a'} \simeq \frac{-4w_0\omega_s\bar{r}_3(t_0)}{3vT_1w_0^2}. \tag{3.15}$$

From this we get

$$S_a = np\int_0^\infty S_{a'}g(\omega_s)\,d\omega_s = \frac{-4R_0\hbar\omega_0}{3vT_1\mathscr{E}_0}$$

where ω_0 is the centre frequency of the distribution function $g(\omega_s)$, assumed to be even about ω_0. Using (2.5) we get

$$\mathscr{E}_0^2 = \frac{16\pi\hbar\omega_0 Q R_0}{3vT_1}. \tag{3.16}$$

Thus the intensity is proportional to the time-independent average population inversion density R_0 in agreement with the rate equation approach.

In order to calculate the oscillator frequency v we use the fact that if we drop a term $\frac{9}{64}w_0^2$ in the denominators of equations (2.19) which is always permissible since $w_0^2 \ll v^2T_1T_2^{-1}$, we can write $C_{a'}$ in the form

$$C_{a'} = \frac{-3w_0^2S_{a'}}{16vT_2^{-1}} - \frac{1}{(\omega_s^2-v^2)^2+4v^2T_2^{-2}}[2w_0(\omega_s^2-v^2)(\bar{r}_3(t_0)\omega_s + \tfrac{3}{4}T_1vw_0S_{a'}) - \tfrac{3}{2}w_0^2vT_2^{-1}S_{a'}]. \tag{3.17}$$

The result (3.15) for $S_{a'}$ at large intensities allows us to substitute in (3.17) to get

$$C_{a'} = \frac{T_2p\mathscr{E}_0\omega_s\bar{r}_3(t_0)}{4T_1v^2\hbar} - \frac{2p\mathscr{E}_0\bar{r}_3(t_0)\omega_s}{\hbar T_1T_2[(\omega_s^2-v^2)^2+4v^2T_2^{-2}]}$$

so that the in-phase component of polarization is

$$C = np\int_0^\infty C_{a'}g(\omega_s)\,d\omega_s \simeq \frac{p^2\mathscr{E}_0T_2R_0\omega_0}{4T_1v^2\hbar} - \frac{\pi p^2R_0\mathscr{E}_0g(\omega_0)}{2vT_1}. \tag{3.18}$$

From (2.5), (3.16) and (3.18) we obtain

$$v^2 - \Omega^2 = -\frac{3v^2}{16Q}\left(\frac{T_2}{v} - \frac{4\pi g(\omega_0)}{\omega_0}\right)w_0^2 \tag{3.19}$$

where w_0 is, as before, $p\mathscr{E}_0\hbar^{-1}$. Thus the oscillator frequency is separated from the cavity resonance frequency by an amount proportional to the intensity. However for

$T_2 \sim 10^{-11}$ s, $Q \sim 10^7$ and $g(\omega_0) \sim (\Delta\omega_0)^{-1} \sim 10^{-12}$ we have

$$v - \Omega \simeq 10^{-19} w_0^2$$

and v will coincide with Ω for all practical purposes: the frequency pulling is cancelled by 'frequency pushing' effects.

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